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TECHNICAL NOTE

CLASSES OF SOLUTIONS IN THE PROBLEM OF STABILITY ANALYSIS IN BARS WITH VARYING CROSS-SECTION AND AXIAL DISTRIBUTED LOADING

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Abstract—We construct classes of analytical solutions for the linear ordinary differential equation of variable coefficients governing the stability analysis of bars with varying cross-section and axial distributed loading. The analytical solutions obtained by Li *et al.* [(1995) *Int. J. Solids Structures*] for two kinds of variable stiffness and axial loading in such types of bars result as special cases.

1. INTRODUCTION

The problem of linear stability and buckling analysis of a straight prismatic bar due to its own weight was discussed by Greenhill (1881), Jasinsky (1902), Dondorff (1907) and Kármán and Biot (1940) [see also Timoshenko and Gere (1961), p. 101]. On the other hand, the same problem, but based on non-linear geometrical elasticity (third-order theory) was successfully investigated and solved by Panayotounakos and Theocaris (1988).

Recently, in a paper by Li *et al.* (1995), the linear stability analysis of straight bars with varying flexural stiffness and axial distributed loading was discussed. Analytical solutions of the governing ordinary differential equation (ODE) of variable coefficients were obtained for two cases expressing the above flexural stiffness and the axial distributed loading of the bar; namely, the cases when both previous quantities are exponential and power functions.

In this paper a successful attempt is made to present classes of analytical solutions for the general linear ODE of variable coefficients governing the above problem. By means of convenient functional transformations, we succeed in constructing solutions for the case when the function of the axial distributed load $N(x)$ is arbitrary, while the function of the flexural stiffness $EJ(x)$ is expressed by way of convenient functional relations with $N(x)$ and vice versa. The two kinds of solutions presented by Li *et al.* (1995) result as special cases of the analytical solutions obtained herein.

2. ANALYTICAL SOLUTIONS OF THE GENERAL LINEAR ODE

The general linear ODE of variable coefficients governing the bending of a straight cantilever bar of variable cross-section and variable axial distributed loading is given by Li *et al.* (1995)

$$M''(x) - \frac{N'(x)}{N(x)} M'(x) + \frac{N(x)}{EJ(x)} M(x) = C_0 \frac{N'(x)}{N(x)}, \tag{1}$$

where

$$N'(x) = -q(x). \tag{2}$$

In the above two equations $M(x)$ represents the internal bending moment, $N(x)$ is the internal axial force, $q(x)$ denotes the axial distributed loading, while $EJ(x)$ denotes the flexural stiffness of the bar. A prime means differentiation with respect to x, while the axis x coincides with the central axis of the bar. In eqn (1) C_0 is a constant determinable by the equation of the slope of the deflected bar

$$y'(x) = \frac{C_0}{N(x)} + \frac{M'(x)}{N(x)}, \tag{3}$$

if we consider the condition

$$\text{for} \quad x = \bar{x},\ y'(\bar{x}) = 0 \Rightarrow C_0 = -M'(\bar{x}) = -Q(\bar{x}).$$

Here $Q(\bar{x})$ denotes the shear force in the $\bar{x}$ cross-section. Setting

$$M(x) = z(x), \quad N(x) = f(x), \quad EJ(x) = g(x), \tag{4}$$

the linear ODE (1) takes the form

$$z'' - \frac{f'}{f}z' + \frac{f}{g}z = C_0\frac{f'}{f}, \tag{5}$$

where f and g are, in general, smooth functions of x. Furthermore, if $z_1(x)$, $z_2(x)$ are two linearly independent solutions of the homogeneous ODE,

$$z'' - \frac{f'}{f}z' + \frac{f}{g}z = 0, \tag{6}$$

then the general integral of eqn (5) can be obtained by means of the Lagrange method in the form

$$z = c_1 z_1 + c_2 z_2 - C_0 z_1 \int \frac{z_2}{D}\frac{f'}{f}\,\mathrm{d}x + C_0 z_2 \int \frac{z_1}{D}\frac{f'}{f}\,\mathrm{d}x, \tag{7}$$

where

$$D = z_1 z_2' - z_1' z_2, \tag{8}$$

while c_1 and c_2 are integration constants determined by suitable boundary conditions. Therefore, the whole problem is focused on the evaluation of the above two linearly independent solutions of the linear homogeneous ODE (6).

The aim of the present note is to extend the paper by Li *et al.* (1995) by trying to construct analytical solutions of the ODE (6) [or the ODEs (5) and (1)] in the case when the function of the axial distributed load $f(x)$ is arbitrary, while the function of the flexural stiffness is expressed by means of convenient functional relations with respect to $f(x)$ and vice versa. Thus, a very large class of important solutions in engineering practice can be obtained. The results already constructed by Li *et al.* (1995) are special cases. Considering that

$$g(x) = \frac{1}{f(x)G\left(\int f(x)\,\mathrm{d}x\right)}, \tag{9}$$

where G is an arbitrary function of the variable $(\int f(x)\,\mathrm{d}x)$, and introducing the functional transformation

$$z(x) = \eta(\xi), \quad \xi = \int f(x)\,\mathrm{d}x; \quad z' = \dot{\eta} f, \quad z'' = \ddot{\eta} f^2 + \dot{\eta} f', \tag{10}$$

in which a dot means differentiation with respect to the variable ξ, we succeed in transforming the homogeneous ODE (6) to the form

$$\ddot{\eta} + G(\xi)\eta = 0. \tag{11}$$

Consequently, the problem under consideration is referred to the construction of solutions of the ODE (11) under certain forms of the function $G(\xi)$, when the function $f(x)$ is arbitrary.

Based on the previously developed equations, we investigate the following cases which are important in engineering practice.

Case a

$$f(x) = \text{arbitrary}, \quad G(\xi) = \lambda\xi + \mu, \quad g(x) = \frac{1}{\mu f + \lambda f \int f\,\mathrm{d}x}, \tag{12}$$

where λ and μ are arbitrary constants.

In this case the ODE (11) becomes

$$\ddot{\eta} + (\lambda\xi + \mu)\eta = 0, \tag{13}$$

which, by way of the functional transformation

$$\eta(\xi) = p(t), \quad t = \lambda\xi + \mu, \tag{14}$$

is reduced to the equation

$$p'' - (-1/\lambda^2)tp = 0, \tag{15}$$

where prime means differentiation with respect to t.

The last ODE is a typical Bessel differential equation with the general solution [Kamke, 1971, p. 440, type (10)]

$$p(t) = t^{1/2}\left[c_1 J_{1/3}\left(-\frac{2}{3\lambda}t^{3/2}\right) + c_2 Y_{1/3}\left(-\frac{2}{3\lambda}t^{3/2}\right)\right], \tag{16}$$

where $J_{1/3}$ and $Y_{1/3}$ are the Bessel functions of the first and second kind, of order 1/3, while c_1 and c_2 are constants of integration. Based on eqn (16) and replacing the variable t by $(\lambda\xi + \mu)$, we deduce the general solution of the ODE (13).

The differential equation (6), corresponding to eqn (13), for the bending moment $(z = M)$ becomes

$$z'' - \frac{f'}{f}z' + \left(\mu + \lambda \int f\,\mathrm{d}x\right)z = 0, \tag{17}$$

which, by way of eqns (16), (14) and (10), has the following general solution:

$$z(x) = \left\{\mu+\lambda\int f\,\mathrm{d}x\right\}^{1/2}\left\{c_1 J_{1/3}\left[-\frac{2}{3\lambda}\left(\mu+\lambda\int f\,\mathrm{d}x\right)^{3/2}\right]\right.$$
$$\left.+c_2 Y_{1/3}\left[-\frac{2}{3\lambda}\left(\mu+\lambda\int f\,\mathrm{d}x\right)^{3/2}\right]\right\}. \quad (18)$$

In the special case

$$\lambda = 0, \quad (19)$$

the ODE (13) has constant coefficients with the general solutions

$$\eta(\xi) = \begin{cases} c_1 \exp(\sqrt{\mu}\xi)+c_2\exp(-\sqrt{\mu}\xi) & \text{for } \mu > 0 \\ c_1\cos(\sqrt{\mu}\xi)+c_2\sin(\sqrt{\mu}\xi) & \text{for } \mu < 0 \\ c_1\xi+c_2 & \text{for } \mu = 0. \end{cases} \quad (20)$$

The ODE (6) for the bending moment ($z = M$) results in the form

$$z''-\frac{f'}{f}z'+\mu f^2 z = 0, \quad (21)$$

the general solutions of which are given by expressions (20) if, instead of ξ, the quantity $\int f\,\mathrm{d}x$ is introduced.

Case b

$$f(x) = \text{arbitrary}, \quad G(\xi) = \lambda\xi^m\ (m \neq 0), \quad g(x) = \frac{1}{\lambda f\left\{\int f\,\mathrm{d}x\right\}^m}. \quad (22)$$

The ODE (11) becomes

$$\ddot{\eta}+\lambda\xi^m\eta = 0, \quad \text{or} \quad \xi^2\ddot{\eta}+\lambda\xi^{m+2}\eta = 0, \quad (23)$$

which is of Bessel type (Kamke, 1971, p. 440) with general solution

$$\eta(\xi) = \xi^{1/2}[c_1 J_\nu(2\nu\sqrt{\lambda}\xi^{1/2})+c_2 Y_\nu(2\nu\sqrt{\lambda}\xi^{1/2})],$$
$$\nu = 1/(m+2). \quad (24)$$

The ODE corresponding to eqn (23) for the bending moment ($z = M$) is written in the form

$$z''-\frac{f'}{f}z'+\left\{\int f\,\mathrm{d}x\right\}^m z = 0, \quad (25)$$

the general solution of which is given by means of expression (24) if, instead of ξ, the quantity $\int f\,\mathrm{d}x$ is introduced.

In the special case where $m = -2$, the ODE (11) becomes of Euler type

$$\xi^2\ddot{\eta}+\lambda\eta = 0 \quad (26)$$

with the general solutions (Kamke, 1971, p. 401)

$$\eta(\xi) = \begin{cases} c_1\xi^{1/2+s} + c_2\xi^{1/2-s} & \text{for } -4\lambda+1 > 0 \\ c_1\sqrt{\xi} + c_2\sqrt{\xi}\ln\xi & \text{for } -4\lambda+1 = 0 \\ c_1\sqrt{\xi}\cos(s\ln\xi) + c_2\sqrt{\xi}\sin(s\ln\xi) & \text{for } -4\lambda+1 < 0 \end{cases} \quad (27)$$

$$s = (|-4\lambda+1|)^{1/2}/2.$$

Here the ODE (6) for the bending moment takes the form

$$z'' - \frac{f'}{f}z' + \frac{\lambda}{\left\{\int f\,\mathrm{d}x\right\}^2}z = 0 \quad (28)$$

and its general solutions are obtained if in eqns (27) the variable ξ is replaced by the quantity $\int f\,\mathrm{d}x$.

Case c

$$f(x) = \text{arbitrary}, \quad G(\xi) = \begin{cases} a\xi^{c-2} - b^2\xi^{2(c-1)} \\ -(a^2\,\mathrm{e}^{2\xi} + b\,\mathrm{e}^{\xi} + c^2) \end{cases},$$

$$g(x) = \begin{bmatrix} \dfrac{1}{f\left\{a\left(\int f\,\mathrm{d}x\right)^{c-2} - b^2\left(\int f\,\mathrm{d}x\right)^{2(c-1)}\right\}} \\ \dfrac{1}{-\left\{a^2\exp\left(2\int f\,\mathrm{d}x\right) + b\exp\left(\int f\,\mathrm{d}x\right) + c^2\right\}} \end{bmatrix}. \quad (29)$$

a, b and c are arbitrary constants.

In both these cases the ODE (11) becomes of the Whittaker type with the general solutions [Kamke, 1971, p. 476, types (12) and (14); p. 427]

$$\eta(\xi) = \xi^{(1-c)/2}[c_1\Phi(a/2bc,\ 1/2c;\ 2b\xi^c/c) + c_2(2b\xi^c/c)^{1-(1/2c)}\Phi[(a/2bc)-(1/2c)+1,\ 2-(1/2c);\ 2b\xi^c/c] \quad (30)$$

and

$$\eta(\xi) = \exp(-\xi/2)[c_1\Phi(-b/2a,\ c,\ 2a\exp(\xi)) + c_2(2a\exp(\xi))^{1-c}\Phi[(-b/2a)-c+1,\ -c;\ 2a\exp(\xi)],$$

respectively.

In eqns (30) $\Phi(a,\gamma;z)$ represents the Φ-function given by Gradshteyn and Ryzhik (1965, p. 1058)

$$\Phi(a,\gamma;z) = 1 + \frac{a}{\gamma}\frac{z}{1!} + \frac{a(a+1)}{\gamma(\gamma+1)}\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{\gamma(\gamma+1)(\gamma+2)}\frac{z^3}{3!} + \cdots.$$

The corresponding general solutions for the bending moment ($z = M$) are obtained by expressions (30) if, instead of ξ, the quantity $\int f\,\mathrm{d}x$ is introduced.

Case d

$$f(x) = \text{arbitrary}, \quad G(\xi) = -F^2(\xi) - \dot{F}(\xi)(F = \text{arbitrary}), \quad g(x) = -\frac{1}{f(F^2+\dot{F})}. \tag{31}$$

In this case, by the functional transformation

$$\eta(\xi) = p(\xi)\exp\left[\int F(\xi)\,\mathrm{d}\xi\right] \tag{32}$$

we succeed in reducing the ODE (11) to the integrable form

$$\ddot{p} + 2F\dot{p} = 0. \tag{33}$$

This differential equation can be directly integrated giving the general solution

$$p(\xi) = c_1 + c_2\int \exp\left[-2\int F\,\mathrm{d}\xi\right]\mathrm{d}\xi, \tag{34}$$

while by means of eqns (32), (31) and (10) the general solution of the ODE (6) becomes

$$z(x) = c_1 \exp\left[\int F\left[\int f\,\mathrm{d}x\right] f\,\mathrm{d}x\right] + c_2 \exp\left[\int F\left[\int f\,\mathrm{d}x\right] f\,\mathrm{d}x\right]\int \exp\left[-2F\left[\int f\,\mathrm{d}x\right] f\,\mathrm{d}x\right] f\,\mathrm{d}x. \tag{35}$$

3. SPECIAL CASES

In this section we shall try to derive the analytical solutions obtained by Li *et al.* (1995) as special cases of one of the four cases examined. This will be achieved by defining the form of the function $f(x)$ expressing the applied axial distributed loading on the bar.

First application

We assume that the axial distributed load is an exponential function, namely

$$n(x) = f(x) = a\exp(-bx/l), \quad b > 0, \quad a > 0, \tag{36}$$

where l denotes the total length of the bar.

Recalling case b of the previous section and using relations (22) we derive

$$EJ(x) = g(x) = \frac{(-b/l)^m}{\lambda a^{m-1}}\,\frac{1}{\exp[-b(m+1)x/l]}. \tag{37}$$

Setting

$$\alpha = -(l/b)^m/\lambda a^{m+1}, \quad \beta = -b(m+1), \tag{38}$$

expression (37) for the flexural stiffness becomes

$$EJ(x) = g(x) = \alpha \exp(-\beta x/l), \tag{39}$$

which is coincident with the corresponding expression defined arbitrarily by Li *et al.* (1995). Then, using eqn (24), the solution for the bending moment $z = M$ results in

$$z(x) = e^{-bx/2l}[c_1 J_v(\overset{*}{\lambda} e^{-bx/2l}) + c_2 Y_v(\overset{*}{\lambda} e^{-bx/2l})], \tag{40}$$

where

$$\overset{*}{\lambda} = 2v\sqrt{\lambda}(-l/b)^{1/2v}. \tag{41}$$

Since, according to eqn (24), we have $1/v = m+2$ $(m \neq 0)$, by way of the second of eqns (38) we derive

$$1/v = -(\beta/b)+1, \quad v = b/(b-\beta)$$

and the final expression for the function z becomes

$$z(x) = e^{cvx/2l}[c_1 J_v(\overset{*}{\lambda} e^{xc/2l}) + c_2 Y_v(\overset{*}{\lambda} e^{xc/2l})]; \quad c = l-b, \tag{42}$$

which coincides with the corresponding solution of Li *et al.* (1995).

Second application

We assume that the axial distributed load is a power function, namely

$$N(x) = f(x) = \alpha(1+\beta x)^c, \tag{43}$$

where α and c are suitable parameters.

Recalling again case b of the previous section and using eqns (22) we derive

$$EJ(x) = g(x) = -\frac{[\beta(1+c)]^m}{\lambda \alpha^{m+1}} \frac{1}{(1+\beta x)^{c+m(c+1)}}. \tag{44}$$

For $m < 0$, expression (44) for the flexural stiffness becomes

$$g(x) = a(1+\beta x)^b; \quad a = 1/\lambda[\beta(1+c)]^{1/m} \alpha^{1/|m|}; \quad b = -c+|m|(1+c), \tag{45}$$

which coincides with the corresponding expression defined arbitrarily by Li *et al.* (1995).

Now, using solution (24), we derive the corresponding solution for the bending moment $z = M$ in the form

$$z(x) = (1+\beta x)^{(1-c)/2}\{c_1 J_v[\overset{*}{\lambda}(1+\beta x)^k] + c_2 Y_v[\overset{*}{\lambda}(1+\beta x)^k]\}; \quad k = (c+1)/2v, \tag{46}$$

which coincides with that constructed in the previous paper. We underline here that the special solutions derived by Li *et al.* (1995) are the same as those given in case b for $m = -2$.

From the above analysis it is obvious that, assuming a specific function for the axial force, we can evaluate through the previously developed technique four different functions for the flexural stiffness and vice versa. From all these solutions other factors, such as materialization of the structure or financial factors, etc., may determine the optimum solution.

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